

DYNAMICS OF THE NEMATIC-ISOTROPIC SHARP INTERFACE FOR THE LIQUID CRYSTAL

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ABSTRACT. In this paper, we derive the sharp interface model of the nematic-isotropic phase transition from the Landau-de Gennes theory by using the matched asymptotic expansion method. The model includes the evolution equation of the velocity and director field of the liquid crystal, the sharp interface and Young-Laplace jump condition on the interface.

1. INTRODUCTION

Liquid crystals are a state of matter that have properties between those of a conventional liquid and those of a solid crystal. They possess several phases, for example, the nematic phase at low temperature and isotropic phase (ordinary fluid) at high temperature. In a nematic phase, the rod-like molecules have no positional order, but they self-align to have long-range directional order with their long axes roughly parallel. There is a first order phase transition to the nematic liquid crystal phase at a clearing point T_{NI} . There is also another temperature T^* . For $T < T^*$, the isotropic phase is unstable, and the nucleation of the nematic phases takes places through a phase-ordering kinetic process. For $T^* < T < T_{NI}$, the isotropic phase is metastable. A classical model which predicts such kind of nematic-isotropic phase transition is the hard-rod model proposed by Onsager [20] (see [16] and references therein for rigorous results). In this paper, we are concerned with the later region, in which stable nematic and isotropic phases can coexist.

There are two classical approaches to describe the nematic-isotropic interface. One approach uses the sharp interface model, which involves solving the governing differential equations with matching boundary conditions at a moving interface separating the nematic phase and isotropic phase [4, 5]. Another approach uses the phase-field model [23, 3]. The nematic-isotropic interface is modeled by a smooth transition region of finite width in the phase-field variable (or order parameter) \mathbf{Q} . The two phases correspond to regions in which the values of \mathbf{Q} are those pertaining to the isotropic and nematic phases. In transition region, the order parameter varies continuously between its two equilibrium values.

In this paper, we choose the phase-field approach to study the dynamics of nematic-isotropic sharp interface in the framework of Landau-de Gennes theory [6, 19]. In this theory, the state of the nematic liquid crystals is described by the macroscopic \mathbf{Q} -tensor order parameter, which is a symmetric, traceless 3×3 matrix. Physically, it can be interpreted as the second-order moment of the orientational distribution function f , that is,

$$\mathbf{Q} = \int_{\mathbb{S}^2} (\mathbf{m}\mathbf{m} - \frac{1}{3}\mathbf{I}) f \, d\mathbf{m}.$$

When $\mathbf{Q} = 0$, the nematic liquid crystal is said to be isotropic. When \mathbf{Q} has two equal non-zero eigenvalues, it is said to be uniaxial and \mathbf{Q} can be written as

$$\mathbf{Q} = s(\mathbf{n}\mathbf{n} - \frac{1}{3}\mathbf{I}), \quad \mathbf{n} \in \mathbb{S}^2.$$

When \mathbf{Q} has three distinct eigenvalues, it is said to be biaxial and \mathbf{Q} can be written as

$$\mathbf{Q} = s(\mathbf{nn} - \frac{1}{3}\mathbf{I}) + \lambda(\mathbf{n}'\mathbf{n}' - \frac{1}{3}\mathbf{I}), \quad \mathbf{n}, \mathbf{n}' \in \mathbb{S}^2, \quad \mathbf{n} \cdot \mathbf{n}' = 0.$$

The general Landau-de Gennes energy functional takes the form

$$\begin{aligned} \mathcal{F}(\mathbf{Q}, \nabla \mathbf{Q}) = & \int_{\Omega} \left\{ \underbrace{\frac{a}{2}\text{Tr}\mathbf{Q}^2 - \frac{b}{3}\text{Tr}\mathbf{Q}^3 + \frac{c}{4}(\text{Tr}\mathbf{Q}^2)^2}_{F_b: \text{bulk energy}} \right. \\ & \left. + \frac{1}{2} \underbrace{\left(L_1 |\nabla \mathbf{Q}|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ij} Q_{kl,i} Q_{kl,j} \right)}_{F_e: \text{elastic energy}} \right\} d\mathbf{x}. \end{aligned} \quad (1.1)$$

Here Ω is a domain in \mathbb{R}^3 , a, b, c are material-dependent and temperature-dependent nonnegative constants and $L_i (i = 1, 2, 3, 4)$ are material dependent elastic constants. We refer to [6, 19] for more introduction.

Since the elastic constants $L_i (i = 1, 2, 3, 4)$ are typically very small compared with a, b, c , we may introduce a small parameter ε in (1.1):

$$\begin{aligned} \mathcal{F}^\varepsilon(\mathbf{Q}, \nabla \mathbf{Q}) = & \frac{1}{\varepsilon^2} \int_{\Omega} \left(\underbrace{\frac{a}{2}\text{Tr}\mathbf{Q}^2 - \frac{b}{3}\text{Tr}\mathbf{Q}^3 + \frac{c}{4}(\text{Tr}\mathbf{Q}^2)^2}_{F_b(\mathbf{Q})} \right) d\mathbf{x} \\ & + \int_{\Omega} \frac{1}{2} \underbrace{\left(L_1 |\nabla \mathbf{Q}|^2 + L_2 Q_{ij,j} Q_{ik,k} + L_3 Q_{ij,k} Q_{ik,j} + L_4 Q_{ij} Q_{kl,i} Q_{kl,j} \right)}_{F_e(\mathbf{Q})} d\mathbf{x}. \end{aligned} \quad (1.2)$$

If \mathbf{Q} is uniaxial everywhere, i.e., $\mathbf{Q} = s(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ with $\mathbf{n} \in \mathbb{S}^2$, then it is easy to see

$$F_b(\mathbf{Q}) = \frac{s^2}{27}(9a - 2bs + 3cs^2),$$

which has double well if and only if $b^2 = 27ac$. As the term $L_4 Q_{ij} Q_{kl,i} Q_{kl,j}$ may cause the energy to be not bounded from below, we take $L_4 = 0$ in the sequel.

There are several dynamic Q-tensor models to describe the flow of the nematic liquid crystal, which are either derived from the molecular kinetic theory for the rigid rods by various closure approximations such as [9, 10], or directly derived by variational method such as Beris-Edwards model [2] and Qian-Sheng's model [24]. In [17], we introduce a systematic schema to derive the continuum static and dynamic model from the molecular kinetic theory, where we derive a dynamic Q-tensor model preserving the basic energy dissipation and the physical range of eigenvalues. In this work, we will choose more popular Beris-Edwards model

$$\mathbf{v}_t^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{v}^\varepsilon = -\nabla \mathbf{p}^\varepsilon + \nabla \cdot (\sigma_\varepsilon^s + \sigma_\varepsilon^a + \sigma_\varepsilon^d), \quad (1.3)$$

$$\nabla \cdot \mathbf{v}^\varepsilon = 0, \quad (1.4)$$

$$\mathbf{Q}_t^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla \mathbf{Q}^\varepsilon + \mathbf{Q}^\varepsilon \cdot \boldsymbol{\Omega}^\varepsilon - \boldsymbol{\Omega}^\varepsilon \cdot \mathbf{Q}^\varepsilon = \frac{1}{\Gamma} \mathbf{H}^\varepsilon + \mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{D}^\varepsilon). \quad (1.5)$$

Here \mathbf{v}^ε is the velocity of the fluid, \mathbf{p}^ε is the pressure, Γ is a collective rotational diffusion constant, $\mathbf{D}^\varepsilon = \frac{1}{2}(\nabla \mathbf{v}^\varepsilon + (\nabla \mathbf{v}^\varepsilon)^T)$, $\boldsymbol{\Omega}^\varepsilon = \frac{1}{2}(\nabla \mathbf{v}^\varepsilon - (\nabla \mathbf{v}^\varepsilon)^T)$, $\sigma_\varepsilon^s, \sigma_\varepsilon^a$ and σ_ε^d are symmetry viscous stress, anti-symmetry viscous stress and distortion stress respectively defined by

$$\sigma_\varepsilon^s = 2\nu \mathbf{D}^\varepsilon - \mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{H}^\varepsilon), \quad \sigma_\varepsilon^a = \mathbf{Q}^\varepsilon \cdot \mathbf{H}^\varepsilon - \mathbf{H}^\varepsilon \cdot \mathbf{Q}^\varepsilon, \quad (\sigma_\varepsilon^d)_{ij} = -\frac{\partial \mathcal{F}^\varepsilon}{\partial Q_{kl,j}} Q_{kl,i}^\varepsilon, \quad (1.6)$$

where $\nu > 0$ is the viscous coefficient and \mathbf{H}^ε is the molecular field defined by

$$\begin{aligned}\mathbf{H}^\varepsilon(\mathbf{Q}^\varepsilon) &= \frac{\delta \mathcal{F}^\varepsilon}{\delta \mathbf{Q}} = -\frac{1}{\varepsilon^2} \frac{\partial F_b}{\partial \mathbf{Q}} + \partial_i \left(\frac{\partial F_e}{\partial \mathbf{Q}_{,i}} \right), \\ &= -\frac{1}{\varepsilon^2} f(\mathbf{Q}) - \mathcal{L}\mathbf{Q},\end{aligned}\tag{1.7}$$

where $f(\mathbf{Q}) = a\mathbf{Q} - b\mathbf{Q}^2 + c|\mathbf{Q}|^2\mathbf{Q} + \frac{b}{3}|\mathbf{Q}|^2\mathbf{I}$ and

$$(\mathcal{L}\mathbf{Q})_{kl} = L_1 \Delta Q_{kl} + \frac{1}{2}(L_2 + L_3)(Q_{km,ml} + Q_{lm,mk} - \frac{2}{3}\delta_{kl}Q_{ij,ij}).$$

And $\mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{M})$ is defined by

$$\mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{M}) = \xi \left(\mathbf{M} \cdot (\mathbf{Q}^\varepsilon + \frac{1}{3}\mathbf{I}) + (\mathbf{Q}^\varepsilon + \frac{1}{3}\mathbf{I}) \cdot \mathbf{M} - 2(\mathbf{Q}^\varepsilon + \frac{1}{3}\mathbf{I})\mathbf{M} : \mathbf{Q}^\varepsilon \right)\tag{1.8}$$

for symmetric and traceless matrix \mathbf{M} , where ξ is a constant depending on the molecular details of a given liquid crystal.

To neglect the boundary effect, we consider the case of the domain $\Omega = \mathbb{T}^3$. For the simplicity of notations, we take $\nu = \Gamma = 1$ in the sequel.

2. SHARP INTERFACE MODEL

We will use the matched asymptotic expansion method motivated by [1] to study the behaviour of the solution $(\mathbf{v}^\varepsilon, \mathbf{Q}^\varepsilon)$ of the system (1.3)–(1.5) when ε is small. The idea is to expand the solution in powers of ε away from the transition region (outer expansion) and inside the transition region (inner expansion). By substituting these expansions into the equations and matching powers of ε , one can determine the limit equation. By matching the inner and outer expansions on the boundaries of the transition region, one can derive the jump condition on the sharp interface.

Now we present a sketch of our main results. Assume that there exists a transition region of width ε separating two domains $\Omega^\pm(t)$. Let $\Gamma(t)$ be a smooth surface centered in the transition region and φ be the signed distance to the sharp interface.

- In the region $\Omega^\pm(t)$, the solution has the expansion in ε :

$$\begin{aligned}\mathbf{v}^\varepsilon(t, x) &= \mathbf{v}_\pm^{(0)}(\tau, t, x) + \varepsilon \mathbf{v}_\pm^{(1)}(\tau, t, x) + \cdots, \\ \mathbf{p}^\varepsilon(t, x) &= \varepsilon^{-1} \mathbf{p}_\pm^{(-1)}(\tau, t, x) + \mathbf{p}_\pm^{(0)}(\tau, t, x) + \cdots, \\ \mathbf{Q}^\varepsilon(t, x) &= \mathbf{Q}_\pm^{(0)}(\tau, t, x) + \varepsilon \mathbf{Q}_\pm^{(1)}(\tau, t, x) + \cdots.\end{aligned}$$

where $\tau = \frac{t}{\varepsilon}$, $\mathbf{Q}^{(0)} = s_\pm(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ with $s_+ = \frac{b+\sqrt{b^2-24ac}}{4c}$ ($= \frac{b}{3c} = \frac{9a}{b}$) and $s_- = 0$. The leading order term $\mathbf{p}^{(-1)}$ of the pressure is harmonic in $\Omega^\pm(t)$. When $\tau \rightarrow +\infty$, $(\mathbf{v}_+^{(0)}, \mathbf{p}_+^{(0)}, \mathbf{n})$ satisfies the Ericksen-Leslie system in $\Omega^+(t)$:

$$\begin{aligned}\partial_t \mathbf{v}_+^{(0)} + \mathbf{v}_+^{(0)} \cdot \nabla \mathbf{v}_+^{(0)} &= -\nabla \mathbf{p}_+^{(0)} + \nabla \cdot (\sigma^L + \sigma^E), \\ \nabla \cdot \mathbf{v}_+^{(0)} &= 0, \\ \mathbf{n} \times (-\Delta \mathbf{n} + \mathbf{N} - \mathbf{D}_+^{(0)} \cdot \mathbf{n}) &= 0.\end{aligned}$$

Here σ^L is the Leslie stress and σ^E is the Ericksen stress (see section 5.1). While, $(\mathbf{v}_-^{(0)}, \mathbf{p}_-^{(0)})$ satisfies the incompressible Navier-Stokes equations in $\Omega^-(t)$:

$$\begin{aligned}\partial_t \mathbf{v}_-^{(0)} + \mathbf{v}_-^{(0)} \cdot \nabla \mathbf{v}_-^{(0)} &= -\nabla \mathbf{p}_-^{(0)} + \Delta \mathbf{v}_-^{(0)}, \\ \nabla \cdot \mathbf{v}_-^{(0)} &= 0.\end{aligned}$$

- In the transition region, the solution has the expansion in ε :

$$\begin{aligned}\mathbf{v}^\varepsilon(t, x, z) &= \tilde{\mathbf{v}}^{(0)}(t, x, z) + \varepsilon \tilde{\mathbf{v}}^{(1)}(t, x, z) + \cdots, \\ \mathbf{Q}^\varepsilon(t, x, z) &= \tilde{\mathbf{Q}}^{(0)}(t, x, z) + \varepsilon \tilde{\mathbf{Q}}^{(1)}(t, x, z) + \cdots,\end{aligned}$$

with $z = \frac{\varphi(t, x)}{\varepsilon}$, where $\tilde{\mathbf{Q}}^{(0)}$ satisfies

$$\tilde{\mathbf{Q}}_{zz}^{(0)} - f(\tilde{\mathbf{Q}}^{(0)}) = 0, \quad \tilde{\mathbf{Q}}^{(0)} \rightarrow s_\pm(\mathbf{nn} - \frac{1}{3}\mathbf{I}) \quad z \rightarrow \pm\infty,$$

which has the uniaxial solution $s(z)(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ with $s(z)$ satisfying

$$-s'' + as - \frac{b}{3}s^2 + \frac{2}{3}cs^3 = 0, \quad s(-\infty) = 0, \quad s(+\infty) = s_+.$$

- The sharp interface is described by the transported mean curvature flow:

$$\varphi_t - \Delta\varphi + \mathbf{v}^{(0)} \cdot \nabla\varphi = 0.$$

- Jump conditions on $\Gamma(t)$:

$$\begin{aligned}[\mathbf{v}^{(0)}] &= 0, \quad \nu \cdot \nabla \mathbf{n}|_{\Gamma(t)} = 0, \\ [\mathbf{p}^{(-1)}] &= -\frac{2}{3} \int_{-\infty}^{+\infty} |s'(z)|^2 dz H, \quad [\mathbf{p}^{(0)}] = [\langle \sigma^L, \nu \otimes \nu \rangle],\end{aligned}$$

where ν is the unit normal of $\Gamma(t)$, and H is the mean curvature of $\Gamma(t)$.

3. CRITICAL POINTS OF THE BULK ENERGY

A matrix \mathbf{Q}_0 is called a critical point of the bulk energy $F_b(\mathbf{Q})$ if $f(\mathbf{Q}_0) = 0$. We have the following characterization for critical points [26].

Proposition 3.1. $f(\mathbf{Q}) = 0$ if and only if

$$\mathbf{Q} = s(\mathbf{nn} - \frac{1}{3}\mathbf{I}),$$

for some $\mathbf{n} \in \mathbb{S}^2$, where $s = 0$ or a solution of $2cs^2 - bs + 3a = 0$, that is,

$$s_1 = \frac{b + \sqrt{b^2 - 24ac}}{4c}, \quad s_2 = \frac{b - \sqrt{b^2 - 24ac}}{4c}.$$

Moreover, the critical point $\mathbf{Q}_0 = s_1(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ is stable.

Given a critical point \mathbf{Q}_0 , the linearized operator $f'(\mathbf{Q}_0)$ of $f(\mathbf{Q})$ around \mathbf{Q}_0 is given by

$$f'(\mathbf{Q}_0)\mathbf{Q} = a\mathbf{Q} - b(\mathbf{Q}_0 \cdot \mathbf{Q} + \mathbf{Q} \cdot \mathbf{Q}_0) + c|\mathbf{Q}_0|^2\mathbf{Q} + 2(\mathbf{Q}_0 : \mathbf{Q})(c\mathbf{Q}_0 + \frac{b}{3}\mathbf{I}). \quad (3.1)$$

It is easy to compute that

$$\begin{aligned}\langle f''(\mathbf{Q}_0)\mathbf{Q}_1, \mathbf{Q}_2 \rangle &= \langle f''(\mathbf{Q}_0)\mathbf{Q}_2, \mathbf{Q}_1 \rangle = -b(\mathbf{Q}_1 \cdot \mathbf{Q}_2 + \mathbf{Q}_2 \cdot \mathbf{Q}_1) + 2c(\mathbf{Q}_0 : \mathbf{Q}_2)\mathbf{Q}_1 \\ &\quad + 2c(\mathbf{Q}_0 : \mathbf{Q}_1)\mathbf{Q}_2 + 2(\mathbf{Q}_1 : \mathbf{Q}_2)(c\mathbf{Q}_0 + \frac{b}{3}\mathbf{I}).\end{aligned} \quad (3.2)$$

In view of Proposition 2.2 in [26], we know that

Proposition 3.2. Let $\mathbf{Q}_0 = s(\mathbf{nn} - \frac{1}{3}\mathbf{I})$ be a critical point with $s \neq 0$. Then the kernel space of the linearized operator $f'(\mathbf{Q}_0)$ is given by

$$\ker f'(\mathbf{Q}_0) = \left\{ \mathbf{nn}^\perp + \mathbf{n}^\perp \mathbf{n} : \mathbf{n}^\perp \in \mathbf{V}_\mathbf{n} \right\},$$

where $\mathbf{V}_\mathbf{n} = \{ \mathbf{n}^\perp \in \mathbb{R}^3 : \mathbf{n}^\perp \cdot \mathbf{n} = 0 \}$.

4. DYNAMICS OF SHARP INTERFACE WITHOUT HYDRODYNAMICS

In this section, we consider the system without hydrodynamics. In such case, the system (1.3)–(1.5) is reduced to a gradient system of \mathbf{Q}^ε :

$$\mathbf{Q}_t^\varepsilon = -\frac{\delta \mathcal{F}^\varepsilon}{\delta \mathbf{Q}} = \frac{1}{\varepsilon^2} f(\mathbf{Q}^\varepsilon) + \mathcal{L} \mathbf{Q}^\varepsilon. \quad (4.1)$$

Assume that there exists a transition region of width ε separating two domains $\Omega^\pm(t)$. Let $\Gamma(t)$ be a smooth surface centered in the transition region and $\varphi(t, x)$ be the signed distance to the sharp interface.

4.1. Outer expansion. We make a formal expansion for \mathbf{Q}^ε in $\Omega_\pm(t)$:

$$\mathbf{Q}^\varepsilon(t, x) = \mathbf{Q}_\pm^{(0)}(t, x) + \varepsilon \mathbf{Q}_\pm^{(1)}(t, x) + \varepsilon^2 \mathbf{Q}_\pm^{(2)}(t, x) + \cdots. \quad (4.2)$$

Then by Taylor expansion, we get

$$f(\mathbf{Q}^\varepsilon) = f(\mathbf{Q}_\pm^{(0)}) + \varepsilon f'(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(1)} + \varepsilon^2 \left(f'(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(2)} + \frac{1}{2} \langle f''(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(1)}, \mathbf{Q}_\pm^{(1)} \rangle \right) + O(\varepsilon^3). \quad (4.3)$$

Plugging (4.2) and (4.3) into (4.1), then equating $\varepsilon^k (k = -2, -1, 0)$, we find that

$$f(\mathbf{Q}_\pm^{(0)}) = 0, \quad (4.4)$$

$$f'(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(1)} = 0, \quad (4.5)$$

$$\partial_t \mathbf{Q}_\pm^{(0)} = \mathcal{L} \mathbf{Q}_\pm^{(0)} - f'(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(2)} - \frac{1}{2} \langle f''(\mathbf{Q}_\pm^{(0)}) \mathbf{Q}_\pm^{(1)}, \mathbf{Q}_\pm^{(1)} \rangle. \quad (4.6)$$

By Proposition 3.1, the equation (4.4) ensures that $\mathbf{Q}_\pm^{(0)}(t, x) = s_\pm (\mathbf{n}(t, x) \mathbf{n}(t, x) - \frac{1}{3} \mathbf{I})$ for some $\mathbf{n}(t, x) \in \mathbb{S}^2$. The equation (4.5) tells us that $\mathbf{Q}_+^{(1)} \in \ker f'(\mathbf{Q}_+^{(0)})$. This means by Proposition 3.2 that

$$\mathbf{Q}_+^{(1)}(t, x) = \mathbf{n}(t, x) \mathbf{n}(t, x)^\perp + \mathbf{n}(t, x)^\perp \mathbf{n}(t, x), \quad \mathbf{n}(t, x)^\perp \in \mathbf{V}_\mathbf{n}, \quad (4.7)$$

from which, it follows that

$$\begin{aligned} \frac{1}{2} \langle f''(\mathbf{Q}_+^{(0)}) \mathbf{Q}_+^{(1)}, \mathbf{Q}_+^{(1)} \rangle &= (2s_+ - b) |\mathbf{n}(t, x)^\perp|^2 \mathbf{n}(t, x) \mathbf{n}(t, x) - b \mathbf{n}(t, x)^\perp \mathbf{n}(t, x)^\perp \\ &\quad + \frac{2}{3} (b - s_+) |\mathbf{n}(t, x)^\perp|^2 \mathbf{I} \perp \ker f'(\mathbf{Q}_+^{(0)}). \end{aligned} \quad (4.8)$$

The solvability of (4.6) to find $\mathbf{Q}_+^{(2)}$ requires that

$$\partial_t \mathbf{Q}_+^{(0)} - \mathcal{L} \mathbf{Q}_+^{(0)} + \frac{1}{2} \langle f''(\mathbf{Q}_+^{(0)}) \mathbf{Q}_+^{(1)}, \mathbf{Q}_+^{(1)} \rangle \perp \ker f'(\mathbf{Q}_+^{(0)}). \quad (4.9)$$

Thanks to (4.8), it is enough to require that

$$\partial_t \mathbf{Q}_+^{(0)} - \mathcal{L} \mathbf{Q}_+^{(0)} \perp \ker f'(\mathbf{Q}_+^{(0)}).$$

This means that

$$(\partial_t \mathbf{Q}_+^{(0)} - \mathcal{L} \mathbf{Q}_+^{(0)}) : (\mathbf{n} \mathbf{n}^\perp + \mathbf{n}^\perp \mathbf{n}) = 0$$

for any $\mathbf{n}^\perp \in \mathbf{V}_\mathbf{n}$. From the proof of Lemma 3.2 in [26], we know that

$$\begin{aligned} \partial_t \mathbf{Q}_+^{(0)} : (\mathbf{n} \mathbf{n}^\perp + \mathbf{n}^\perp \mathbf{n}) &= 2s_+ \mathbf{n}_t \cdot \mathbf{n}^\perp, \\ -\mathcal{L} \mathbf{Q}_+^{(0)} : (\mathbf{n} \mathbf{n}^\perp + \mathbf{n}^\perp \mathbf{n}) &= \frac{1}{s_+} \mathbf{h} \cdot \mathbf{n}^\perp, \end{aligned}$$

where $\mathbf{h} = -\frac{\delta E(\mathbf{n}, \nabla \mathbf{n})}{\delta \mathbf{n}}$ with $E(\mathbf{n}, \nabla \mathbf{n})$ the Oseen-Frank energy defined by

$$E = \frac{k_1}{2}(\nabla \cdot \mathbf{n})^2 + \frac{k_2}{2}(\mathbf{n} \cdot (\nabla \times \mathbf{n}))^2 + \frac{k_3}{2}|\mathbf{n} \times (\nabla \times \mathbf{n})|^2 + \frac{k_2 + k_4}{2}(\text{tr}(\nabla \mathbf{n})^2 - (\nabla \cdot \mathbf{n})^2), \quad (4.10)$$

where the elastic constants k_1, k_2, k_3, k_4 are given by

$$k_1 = k_3 = (2L_1 + L_2 + L_3)(s_+)^2, \quad k_2 = 2L_1(s_+)^2, \quad k_4 = L_3(s_+)^2. \quad (4.11)$$

Hence, we conclude that \mathbf{n} satisfies

$$(2(s_+)^2 \mathbf{n}_t + \mathbf{h}) \times \mathbf{n} = 0. \quad (4.12)$$

In special case $L_1 = 1, L_2 = L_3 = 0$ (thus, $k_1 = k_2 = k_3 = 2(s_+)^2, k_4 = 0$), we have

$$E = (s_+)^2 |\nabla \mathbf{n}|^2, \quad \mathbf{h} = -2(s_+)^2 \Delta \mathbf{n}.$$

In this case, the equation (4.12) is just the well-known harmonic heat flow.

4.2. Inner expansion. In the transition region, we make the following expansion for \mathbf{Q}^ε :

$$\mathbf{Q}^\varepsilon(t, x) = \tilde{\mathbf{Q}}^{(0)}(t, x, z) + \varepsilon \tilde{\mathbf{Q}}^{(1)}(t, x, z) + \varepsilon^2 \tilde{\mathbf{Q}}^{(2)}(t, x, z) + \cdots, \quad (4.13)$$

where $z = \frac{\varphi(t, x)}{\varepsilon}$. Simple calculations give

$$\begin{aligned} \mathbf{Q}_t^\varepsilon &= \varepsilon^{-1} \varphi_t \tilde{\mathbf{Q}}_z^{(0)} + \varphi_t \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{Q}}_t^{(0)} + O(\varepsilon), \\ \mathcal{L} \mathbf{Q}^\varepsilon &= \varepsilon^{-2} \mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(0)}) + \varepsilon^{-1} \left(\mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(1)}) + \mathcal{B}_1(\nabla \varphi, \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) + \mathcal{B}_2(\nabla^2 \varphi, \tilde{\mathbf{Q}}_z^{(0)}) \right) + O(1), \\ f(\mathbf{Q}^\varepsilon) &= f(\tilde{\mathbf{Q}}^{(0)}) + \varepsilon f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} + \varepsilon^2 (f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(2)} + \frac{1}{2} \langle f''(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(1)} \rangle) + O(\varepsilon^3), \end{aligned}$$

where

$$\begin{aligned} (\mathcal{A}(\nabla \varphi, \mathbf{Q}))_{kl} &= L_1 Q_{kl} |\nabla \varphi|^2 + \frac{1}{2} (L_2 + L_3) (Q_{km} \partial_m \varphi \partial_l \varphi + Q_{lm} \partial_m \varphi \partial_k \varphi - \frac{2}{3} \delta_{kl} Q_{ij} \partial_i \varphi \partial_j \varphi), \\ (\mathcal{B}_1(\nabla \varphi, \nabla_x \mathbf{Q}))_{kl} &= 2L_1 \partial_i Q_{kl} \partial_i \varphi + \frac{1}{2} (L_2 + L_3) (\partial_m Q_{km} \partial_l \varphi + \partial_l Q_{km} \partial_m \varphi + \partial_m Q_{lm} \partial_k \varphi \\ &\quad + \partial_k Q_{lm} \partial_m \varphi - \frac{2}{3} \delta_{kl} \partial_i Q_{ij} \partial_j \varphi - \frac{2}{3} \delta_{kl} \partial_j Q_{ij} \partial_i \varphi), \\ (\mathcal{B}_2(\nabla^2 \varphi, \mathbf{Q}))_{kl} &= L_1 Q_{kl} \Delta \varphi + \frac{1}{2} (L_2 + L_3) (Q_{km} \partial_m \partial_l \varphi + Q_{lm} \partial_m \partial_k \varphi - \frac{2}{3} \delta_{kl} Q_{ij} \partial_i \partial_j \varphi). \end{aligned}$$

Substituting the above expansion into (4.1), then equating $\varepsilon^{-2}, \varepsilon^{-1}$ terms, we obtain

$$-\mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(0)}) + f(\tilde{\mathbf{Q}}^{(0)}) = 0, \quad (4.14)$$

$$-\mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(1)}) + f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} = -\varphi_t \tilde{\mathbf{Q}}_z^{(0)} + \mathcal{B}_1(\nabla \varphi, \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) + \mathcal{B}_2(\nabla^2 \varphi, \tilde{\mathbf{Q}}_z^{(0)}). \quad (4.15)$$

Now let us derive the evolution equation of sharp interface $\varphi(t, x)$. Assume that

$$\tilde{\mathbf{Q}}^{(0)}(t, x, z) \longrightarrow \mathbf{Q}_\pm^{(0)}(t, x) \quad \text{as } z \rightarrow \pm\infty.$$

Then by integration by parts and (4.14), we find that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \left(-\mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(1)}) + f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} \right) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &= \int_{-\infty}^{+\infty} \left(-\mathcal{A}(\nabla \varphi, \tilde{\mathbf{Q}}_{zz}^{(0)}) + f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} \right) : \tilde{\mathbf{Q}}_z^{(1)} dz = 0. \end{aligned} \quad (4.16)$$

Here we used the fact that $\tilde{\mathbf{Q}}^{(0)}, \tilde{\mathbf{Q}}^{(1)}$ are traceless and $\mathbf{F}_b(\mathbf{Q}_\pm^{(0)}) = 0$. On the other hand, we have

$$\begin{aligned} & \int_{-\infty}^{+\infty} \left(-\varphi_t \tilde{\mathbf{Q}}_z^{(0)} + \mathcal{B}_1(\nabla\varphi, \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) + \mathcal{B}_2(\nabla^2\varphi, \tilde{\mathbf{Q}}_z^{(0)}) \right) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &= -c\varphi_t + \nabla \cdot (A\nabla\varphi), \end{aligned}$$

where

$$\begin{aligned} c(t, x) &= \int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}(t, x, z)|^2 dz, \\ A_{kl}(t, x) &= L_1 c(t, x) \delta_{kl} + (L_2 + L_3) \int_{-\infty}^{+\infty} \tilde{\mathbf{Q}}_{km,z}^{(0)}(t, x, z) \tilde{\mathbf{Q}}_{ml,z}^{(0)}(t, x, z) dz. \end{aligned}$$

This combined with (4.15) and (4.16) gives

$$c\varphi_t - \nabla \cdot (A\nabla\varphi) = 0. \quad (4.17)$$

In special case $L_1 = 1, L_2 = L_3 = 0$, we have

$$c\varphi_t - \nabla \cdot (c\nabla\varphi) = 0, \quad (4.18)$$

and $\tilde{\mathbf{Q}}^{(0)}$ satisfies

$$-\tilde{\mathbf{Q}}_{zz}^{(0)} + f(\tilde{\mathbf{Q}}^{(0)}) = 0 \quad (4.19)$$

together with the boundary conditions

$$\begin{aligned} \tilde{\mathbf{Q}}^{(0)}(t, x, z) &\longrightarrow s_+ (\mathbf{n}(t, x) \mathbf{n}(t, x) - \frac{1}{3} \mathbf{I}) \quad \text{as } z \rightarrow +\infty, \\ \tilde{\mathbf{Q}}^{(0)}(t, x, z) &\longrightarrow 0 \quad \text{as } z \rightarrow -\infty. \end{aligned}$$

If the solution of (4.19) is uniaxial with the form $s(z)(\mathbf{n}(t, x) \mathbf{n}(t, x) - \frac{1}{3} \mathbf{I})$, then s should satisfy

$$-s'' + as - \frac{b}{3}s^2 + \frac{2}{3}cs^3 = 0, \quad s(-\infty) = 0, \quad s(+\infty) = s_+. \quad (4.20)$$

Therefore, the function $c(t, x)$ is independent of (t, x) , and the equation (4.18) is reduced to

$$\varphi_t - \Delta\varphi = 0,$$

which is the well-known mean curvature flow. Thus, the equation (4.15) is reduced to

$$-\tilde{\mathbf{Q}}_{zz}^{(1)} + f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} = 2\nabla\varphi \cdot \nabla \tilde{\mathbf{Q}}_z^{(0)}.$$

Note that it holds for any $\tilde{\mathbf{T}} = s(z)(\mathbf{nn}' + \mathbf{n}'\mathbf{n})$ with $\mathbf{n}' \perp \mathbf{n}$,

$$-\tilde{\mathbf{T}}_{zz} + \langle f'(\tilde{\mathbf{Q}}^{(0)}), \tilde{\mathbf{T}} \rangle = 0,$$

which implies that

$$\int_{-\infty}^{+\infty} \nabla\varphi \cdot \nabla \partial_z [s(z)(\mathbf{nn} - \frac{1}{3}\mathbf{I})] : s(z)(\mathbf{nn}' + \mathbf{n}'\mathbf{n}) dz = 0.$$

While, this is equivalent to

$$\nabla\varphi \cdot \nabla(\mathbf{nn} - \frac{1}{3}\mathbf{I}) : (\mathbf{nn}' + \mathbf{n}'\mathbf{n}) = 0,$$

that is,

$$(\nabla\varphi \cdot \nabla\mathbf{n}) \cdot \mathbf{n}' = 0, \quad \text{for all } \mathbf{n}' \perp \mathbf{n}.$$

On the other hand, we have

$$(\nabla\varphi \cdot \nabla\mathbf{n}) \cdot \mathbf{n} = 0.$$

Especially, this means that \mathbf{n} should satisfy the Neumann condition on the sharp interface

$$\nu \cdot \nabla \mathbf{n} = 0 \quad \text{on} \quad \Gamma(t). \quad (4.21)$$

Here ν is the unit normal of Γ .

In summary, we derive the sharp interface model without hydrodynamics from the gradient system (4.1):

$$\begin{aligned} \mathbf{n}_t - \Delta \mathbf{n} &= |\nabla \mathbf{n}|^2 \mathbf{n} \quad \text{in} \quad \Omega^+(t), \\ \nu \cdot \nabla \mathbf{n} &= 0 \quad \text{on} \quad \Gamma(t). \end{aligned}$$

While, the sharp interface $\Gamma(t)$ is determined by the mean curvature flow

$$\varphi_t - \Delta \varphi = 0.$$

4.3. Asymptotic analysis of Landau-de Gennes energy. Assume that the leading order term of \mathbf{Q}^ε is uniaxial with the form

$$\mathbf{Q}^\varepsilon(t, x) \sim s\left(\frac{\varphi(t, x)}{\varepsilon}\right) (\mathbf{n}(t, x)\mathbf{n}(t, x) - \frac{1}{3}\mathbf{I}) \triangleq \mathbf{Q}_0(t, x).$$

This is a reasonable assumption at least in some special cases (for example, $L_2 = L_3 = 0$ or $L_2 < 0, L_3 = 0$) by the outer-inner asymptotic analysis and [22].

Without loss of generality we consider in the case of $a = \frac{1}{3}, b = 3, c = 1$ and then $s_+ = 1$. And we assume $L_3 = L_4 = 0$. Then Landau-de Gennes energy takes

$$\mathcal{F}^\varepsilon(\mathbf{Q}, \nabla \mathbf{Q}) = \frac{1}{\varepsilon^2} \int_{\mathbb{T}^3} \frac{a}{2} \text{Tr} \mathbf{Q}^2 - \frac{b}{3} \text{Tr} \mathbf{Q}^3 + \frac{c}{4} (\text{Tr} \mathbf{Q}^2)^2 \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{T}^3} L_1 |\nabla \mathbf{Q}|^2 + L_2 Q_{ij,j} Q_{ik,k} \, d\mathbf{x}.$$

Let $S_\varepsilon(x) = s\left(\frac{\varphi(t,x)}{\varepsilon}\right)$. Putting \mathbf{Q}_0 into $\mathcal{F}^\varepsilon(\cdot, \nabla \cdot)$, we deduce that

$$\begin{aligned} \varepsilon \mathcal{F}^\varepsilon(\mathbf{Q}_0, \nabla \mathbf{Q}_0) &= \frac{c}{9\varepsilon} \int_{\mathbb{T}^3} S_\varepsilon^2 (s_+ - S_\varepsilon)^2 \, d\mathbf{x} + \frac{\varepsilon L_1}{2} \int_{\mathbb{T}^3} \left(\frac{2}{3} |\nabla S_\varepsilon|^2 + S_\varepsilon^2 |\nabla \mathbf{n}|^2 \right) \, d\mathbf{x} \\ &\quad + \frac{\varepsilon L_2}{2} \int_{\mathbb{T}^3} \left(\frac{1}{9} |\nabla S_\varepsilon|^2 + \frac{1}{3} (\mathbf{n} \cdot \nabla S_\varepsilon)^2 + S_\varepsilon^2 (|\mathbf{n} \cdot \nabla \mathbf{n}|^2 + (\nabla \cdot \mathbf{n})^2) \right. \\ &\quad \left. + \frac{2}{3} S_\varepsilon (2(\mathbf{n} \cdot \nabla S_\varepsilon)(\nabla \cdot \mathbf{n}) - \nabla S_\varepsilon \cdot (\mathbf{n} \cdot \nabla \mathbf{n})) \right) \, d\mathbf{x} \\ &= \frac{1}{9} \int_{\mathbb{T}^3} \left(\frac{c S_\varepsilon^2 (s_+ - S_\varepsilon)^2}{\varepsilon} + \frac{\varepsilon}{2} (6L_1 + L_2) |\nabla S_\varepsilon|^2 + \frac{3\varepsilon L_2}{2} (\mathbf{n} \cdot \nabla S_\varepsilon)^2 \right) \, d\mathbf{x} \\ &\quad + \frac{\varepsilon}{2} \int_{\mathbb{T}^3} \left(L_1 S_\varepsilon^2 |\nabla \mathbf{n}|^2 + L_2 S_\varepsilon^2 (|\mathbf{n} \cdot \nabla \mathbf{n}|^2 + (\nabla \cdot \mathbf{n})^2) \right) \, d\mathbf{x} \\ &\quad + \frac{\varepsilon L_2}{2} \int_{\mathbb{T}^3} \left(\frac{2}{3} S_\varepsilon [2(\mathbf{n} \cdot \nabla S_\varepsilon)(\nabla \cdot \mathbf{n}) - \nabla S_\varepsilon \cdot (\mathbf{n} \cdot \nabla \mathbf{n})] \right) \, d\mathbf{x} \\ &\triangleq A + B + C. \end{aligned}$$

Direct calculations lead to

$$\begin{aligned}
A &= \frac{1}{9} \int_{\Gamma} \int_{-\infty}^{+\infty} \left(cs^2(s_+ - s)^2 + \frac{1}{2}(6L_1 + L_2)s_z^2 + \frac{3L_2}{2}(\mathbf{n} \cdot \nu)^2 s_z^2 \right) dz d\sigma \\
&= \int_{\Gamma} (\alpha + \beta(\mathbf{n} \cdot \nu)^2) d\sigma, \\
B &= \frac{\varepsilon}{2} \int_{\Omega^+} \left(L_1 |\nabla \mathbf{n}|^2 + L_2 (|\mathbf{n} \cdot \nabla \mathbf{n}|^2 + (\nabla \cdot \mathbf{n})^2) \right) d\mathbf{x} + o(\varepsilon), \\
C &= \frac{\varepsilon L_2}{2} \int_{\Gamma} \int_{-\infty}^{+\infty} \left(\frac{2}{3} s [2(\mathbf{n} \cdot \nu) s_z (\nabla \cdot \mathbf{n}) - s_z \nu \cdot (\mathbf{n} \cdot \nabla \mathbf{n})] \right) dz d\sigma \\
&= \frac{\varepsilon L_2}{6} \int_{\Gamma} \left(2(\mathbf{n} \cdot \nu)(\nabla \cdot \mathbf{n}) - \nu \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) \right) d\sigma,
\end{aligned}$$

where ν is the unit normal of Γ and

$$\begin{aligned}
\alpha &= \frac{1}{9} \int_{-\infty}^{+\infty} \left(cs^2(s_+ - s)^2 + \frac{1}{2}(6L_1 + L_2)s_z^2 \right) dz, \\
\beta &= \frac{3L_2}{18} \int_{-\infty}^{+\infty} s_z^2 dz.
\end{aligned}$$

This gives the following asymptotic of $\mathcal{F}^\varepsilon(\mathbf{Q}, \nabla \mathbf{Q})$ as $\varepsilon \rightarrow 0$,

$$\begin{aligned}
\varepsilon \mathcal{F}^\varepsilon(\mathbf{Q}_0, \nabla \mathbf{Q}_0) &= \int_{\Gamma} (\alpha + \beta(\mathbf{n} \cdot \nu)^2) d\sigma \\
&\quad + \varepsilon \left\{ \frac{1}{2} \int_{\Omega^+} \left(L_1 |\nabla \mathbf{n}|^2 + L_2 (|\mathbf{n} \cdot \nabla \mathbf{n}|^2 + (\nabla \cdot \mathbf{n})^2) \right) d\mathbf{x} \right. \\
&\quad \left. + \frac{L_2}{6} \int_{\Gamma} \left(2(\mathbf{n} \cdot \nu)(\nabla \cdot \mathbf{n}) - \nu \cdot (\mathbf{n} \cdot \nabla \mathbf{n}) \right) d\sigma \right\} + o(\varepsilon), \tag{4.22}
\end{aligned}$$

which is consistent with the total free energy introduced in [25]. The second part on the right hand side corresponds to the well-known Oseen-Frank energy. Let us give some explanations for the other parts from the point of view of energy minimization.

Remark 4.1. We view α as a functional of s , then minimize $\alpha(s)$ in the class $s \in C^2, s(+\infty) = 1, s(-\infty) = 0$. That is,

$$\min_{s \in C^2, s(+\infty)=1, s(-\infty)=0} \frac{1}{9} \int_{-\infty}^{+\infty} \left(cs^2(s_+ - s)^2 + \frac{1}{2}(6L_1 + L_2)s_z^2 \right) dz.$$

If $s(z)$ is a minimizer, it should satisfy the following Euler-Lagrangian equation

$$-(6L_1 + L_2)s'' + 2c(s_+^2 s - 3s_+ s^2 + 2s^3) = 0.$$

This is consistent with (4.20) in the case of $L_1 = 1, L_2 = 0$.

Remark 4.2. For the case of $L_2 < 0$ (hence, $\beta < 0$), we know from (4.22) that \mathbf{n} should take the normal ν on Γ in order that the energy is small as soon as possible. In this case, we have by $\nabla \cdot \mathbf{n} = \nabla_{\Gamma} \cdot \mathbf{n} + \nu \cdot \partial_{\nu} \mathbf{n}$ that

$$\begin{aligned}
\int_{\Gamma} \left(2(\nabla \cdot \mathbf{n}) - \nu \cdot (\partial_{\nu} \mathbf{n}) \right) d\sigma &= \int_{\Gamma} \left(2(\nabla_{\Gamma} \cdot \mathbf{n}) + \nu \cdot (\partial_{\nu} \mathbf{n}) \right) d\sigma \\
&= 4 \int_{\Gamma} H d\sigma,
\end{aligned}$$

where H is the mean curvature of the interface Γ . If $L_2 > 0$, it seems reasonable to conjecture from (4.22) that \mathbf{n} should be tangent to $\Gamma(t)$. However, the above analysis does not work, since the uniaxial solution is unstable in the case of $L_2 > 0$ by numerical analysis [11] and [22].

5. DYNAMICS OF SHARP INTERFACE WITH HYDRODYNAMICS

In this section, we will consider the case of $L_1 = 1, L_2 = L_3 = 0$ in order to simplify the analysis. So, the molecular field $\mathbf{H}^\varepsilon = -\frac{1}{\varepsilon^2}f(\mathbf{Q}) - \Delta\mathbf{Q}$. Again, assume that there exists a transition region of width ε separating two domains $\Omega^\pm(t)$. Let $\Gamma(t)$ be a smooth surface centered in the transition region and $\varphi(t, x)$ be the signed distance to the sharp interface.

5.1. Outer expansion. In order to match strong singularity of the distortion stress σ_ε^d , we need to introduce a fast time scale $\tau = \frac{t}{\varepsilon}$. We make a formal expansion for $(\mathbf{v}^\varepsilon, \mathbf{p}^\varepsilon, \mathbf{Q}^\varepsilon)$ in $\Omega_\pm(t)$:

$$\begin{aligned}\mathbf{v}^\varepsilon(t, x) &= \mathbf{v}_\pm^{(0)}(\tau, t, x) + \varepsilon \mathbf{v}_\pm^{(1)}(\tau, t, x) + \varepsilon^2 \mathbf{v}_\pm^{(2)}(\tau, t, x) + \cdots, \\ \mathbf{p}^\varepsilon(t, x) &= \varepsilon^{-2} \mathbf{p}_\pm^{(-2)}(\tau, t, x) + \varepsilon^{-1} \mathbf{p}_\pm^{(-1)}(\tau, t, x) + \mathbf{p}_\pm^{(0)}(\tau, t, x) + \cdots, \\ \mathbf{Q}^\varepsilon(t, x) &= \mathbf{Q}_\pm^{(0)}(\tau, t, x) + \varepsilon \mathbf{Q}_\pm^{(1)}(\tau, t, x) + \varepsilon^2 \mathbf{Q}_\pm^{(2)}(\tau, t, x) + \cdots.\end{aligned}$$

Simple calculations lead to

$$\begin{aligned}\mathbf{H}^\varepsilon &= -\varepsilon^{-2}f(\mathbf{Q}_\pm^{(0)}) - \varepsilon^{-1}f'(\mathbf{Q}_\pm^{(0)})\mathbf{Q}_\pm^{(1)} + \left(\Delta\mathbf{Q}_\pm^{(0)} - f'(\mathbf{Q}_\pm^{(0)})\mathbf{Q}_\pm^{(2)}\right. \\ &\quad \left. - \frac{1}{2}\langle f''(\mathbf{Q}_\pm^{(0)})\mathbf{Q}_\pm^{(1)}, \mathbf{Q}_\pm^{(1)} \rangle\right) + \cdots \triangleq \varepsilon^{-2}\mathbf{H}_\pm^{(-2)} + \varepsilon^{-1}\mathbf{H}_\pm^{(-1)} + \cdots, \\ \mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{H}^\varepsilon) &= \varepsilon^{-2}\mathbf{S}_{\mathbf{Q}_\pm^{(0)}}(\mathbf{H}_\pm^{(-2)}) + \sum_{m=-1}^0 \varepsilon^m \xi \left(\sum_{k=-2}^m \mathbf{H}_\pm^{(k)} \cdot \mathbf{Q}_\pm^{(m-k)} + \sum_{k=-2}^m \mathbf{Q}_\pm^{(m-k)} \cdot \mathbf{H}_\pm^{(k)} + \frac{2}{3}\mathbf{H}_\pm^{(m)} \right. \\ &\quad \left. - 2 \sum_{k=0}^{m+2} \mathbf{Q}_\pm^{(k)} \sum_{l=-2}^{m-k} \mathbf{H}_\pm^{(l)} : \mathbf{Q}_\pm^{(m-k-l)} - \frac{2}{3}\mathbf{I} \sum_{k=-2}^m \mathbf{H}_\pm^{(k)} : \mathbf{Q}_\pm^{(m-k)} \right) + \cdots \\ &\triangleq \varepsilon^{-2}\mathbf{S}_{\mathbf{H}, \pm}^{(-2)} + \varepsilon^{-1}\mathbf{S}_{\mathbf{H}, \pm}^{(-1)} + \cdots,\end{aligned}$$

and

$$\begin{aligned}\sigma_\varepsilon^s &= -\varepsilon^{-2}\mathbf{S}_{\mathbf{H}, \pm}^{(-2)} - \varepsilon^{-1}\mathbf{S}_{\mathbf{H}, \pm}^{(-1)} + (\mathbf{D}_\pm^{(0)} - \mathbf{S}_{\mathbf{H}, \pm}^{(0)}) + \cdots \triangleq \varepsilon^{-2}\sigma_{(-2), \pm}^s + \varepsilon^{-1}\sigma_{(-1), \pm}^s + \cdots, \\ \sigma_\varepsilon^a &= \sum_{m=-2}^0 \varepsilon^m \left(\sum_{k=-2}^m \mathbf{Q}_\pm^{(m-k)} \cdot \mathbf{H}_\pm^{(k)} - \sum_{k=-2}^m \mathbf{H}_\pm^{(k)} \cdot \mathbf{Q}_\pm^{(m-k)} \right) + \cdots \\ &\triangleq \varepsilon^{-2}\sigma_{(-2), \pm}^a + \varepsilon^{-1}\sigma_{(-1), \pm}^a + \cdots, \\ \sigma_\varepsilon^d &= \sigma^d(\mathbf{Q}_\pm^{(0)}, \mathbf{Q}_\pm^{(0)}) + \cdots \triangleq \sigma_{(0), \pm}^d + \cdots.\end{aligned}$$

Plugging these expansions into (1.3)-(1.5), then equating the $\varepsilon^k (k = -2, -1, 0)$ terms in (1.3), we get

$$\nabla \mathbf{p}_\pm^{(-2)} = \nabla \cdot \sigma_{(-2), \pm}^s + \nabla \cdot \sigma_{(-2), \pm}^a, \quad (5.1)$$

$$\partial_\tau \mathbf{v}_\pm^{(0)} = -\nabla \mathbf{p}_\pm^{(-1)} + \nabla \cdot \sigma_{(-1), \pm}^s + \nabla \cdot \sigma_{(-1), \pm}^a, \quad (5.2)$$

$$\partial_\tau \mathbf{v}_\pm^{(1)} + \partial_t \mathbf{v}_\pm^{(0)} + \mathbf{v}_\pm^{(0)} \cdot \nabla \mathbf{v}_\pm^{(0)} = -\nabla \mathbf{p}_\pm^{(0)} + \nabla \cdot (\sigma_{(0), \pm}^s + \sigma_{(0), \pm}^a + \sigma_{(0), \pm}^d). \quad (5.3)$$

Equating the $\varepsilon^k (k = 0, 1)$ terms in (1.4), we get

$$\nabla \cdot \mathbf{v}_\pm^{(0)} = \nabla \cdot \mathbf{v}_\pm^{(1)} = 0. \quad (5.4)$$

Equating the $\varepsilon^k (k = -2, -1, 0)$ terms in (1.5), we get

$$-f(\mathbf{Q}_\pm^{(0)}) = \mathbf{H}_\pm^{(-2)} = 0, \quad (5.5)$$

$$\partial_\tau \mathbf{Q}_\pm^{(0)} = -f'(\mathbf{Q}_\pm^{(0)})\mathbf{Q}_\pm^{(1)} = \mathbf{H}_\pm^{(-1)}, \quad (5.6)$$

$$\partial_\tau \mathbf{Q}_\pm^{(1)} + \partial_t \mathbf{Q}_\pm^{(0)} + \mathbf{v}_\pm^{(0)} \cdot \nabla \mathbf{Q}_\pm^{(0)} + \mathbf{Q}_\pm^{(0)} \cdot \boldsymbol{\Omega}_\pm^{(0)} - \boldsymbol{\Omega}_\pm^{(0)} \cdot \mathbf{Q}_\pm^{(0)} = \mathbf{H}_\pm^{(0)} + \mathbf{S}_{\mathbf{Q}_\pm^{(0)}}(\mathbf{D}_\pm^{(0)}), \quad (5.7)$$

where $\boldsymbol{\Omega}_\pm^{(0)} = \frac{1}{2}(\nabla \mathbf{v}_\pm^{(0)} - (\nabla \mathbf{v}_\pm^{(0)})^T)$ and $\mathbf{D}_\pm^{(0)} = \frac{1}{2}(\nabla \mathbf{v}_\pm^{(0)} + (\nabla \mathbf{v}_\pm^{(0)})^T)$.

By (5.5), (5.6), (5.1) and (5.2), we have

$$\sigma_{(-2),\pm}^s = 0, \quad \sigma_{(-2),\pm}^a = 0, \quad \nabla \mathbf{p}_\pm^{(-2)} = 0.$$

Similar to analysis in Section 3.1, we know from (5.5) that

$$\mathbf{Q}_\pm^{(0)}(\tau, t, x) = s_\pm(\mathbf{n}(\tau, t, x)\mathbf{n}(\tau, t, x) - \frac{1}{3}\mathbf{I}) \quad (5.8)$$

for some $\mathbf{n}(t, x) \in \mathbb{S}^2$. The solvability of (5.6) to find $\mathbf{Q}_+^{(1)}$ requires that

$$\partial_\tau \mathbf{Q}_+^{(0)} \perp \ker f'(\mathbf{Q}_+^{(0)}). \quad (5.9)$$

By Proposition 3.2, (5.8) and (5.9) require that for any $\mathbf{n}^\perp \in \mathbf{V}_\mathbf{n}$,

$$0 = (\mathbf{n}\mathbf{n}_\tau + \mathbf{n}_\tau\mathbf{n}) : (\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) = 2\mathbf{n}_\tau \cdot \mathbf{n}^\perp,$$

which along with the fact that $\mathbf{n}_\tau \cdot \mathbf{n} = 0$ and (5.6) implies that

$$\mathbf{n}_\tau = 0, \quad \partial_\tau \mathbf{Q}_+^{(0)} = 0, \quad \mathbf{H}_+^{(-1)} = 0.$$

Hence, we have

$$\sigma_{(-1),\pm}^s = 0, \quad \sigma_{(-1),\pm}^a = 0,$$

and

$$\partial_\tau \mathbf{v}_\pm^{(0)} = -\nabla \mathbf{p}^{(-1)}_\pm. \quad (5.10)$$

This implies that $\mathbf{p}^{(-1)}_\pm$ is harmonic by (5.4).

In order to derive the evolution equation of $(\mathbf{v}^{(0)}, \mathbf{Q}^{(0)})$ with respect to time scale t , we assume that as $\tau \rightarrow +\infty$,

$$\begin{aligned} \mathbf{v}_\pm^{(0)}(\tau, t, x) &\rightarrow \mathbf{v}_\pm^{(0)}(\infty, t, x), \quad \mathbf{p}_\pm^{(0)}(\tau, t, x) \rightarrow \mathbf{p}_\pm^{(0)}(\infty, t, x), \\ \mathbf{Q}_+^{(1)}(\tau, t, x) &\rightarrow \mathbf{Q}_+^{(1)}(\infty, t, x), \quad \partial_\tau \mathbf{Q}_+^{(1)}(\tau, t, x) \rightarrow 0. \end{aligned}$$

Without confusing notations, we still use the same notations to denote the corresponding limits as $\tau \rightarrow +\infty$.

Recall that

$$\mathbf{H}_+^{(0)} = \Delta \mathbf{Q}_+^{(0)} - f'(\mathbf{Q}_+^{(0)})\mathbf{Q}_+^{(2)} - \frac{1}{2}\langle f''(\mathbf{Q}_+^{(0)})\mathbf{Q}_+^{(1)}, \mathbf{Q}_\pm^{(1)} \rangle$$

with $\mathbf{Q}_+^{(1)} = \mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}$ for $\mathbf{n}^\perp \in \mathbf{V}_\mathbf{n}$. Then by (4.8), the solvability of (5.7) to find $\mathbf{Q}_+^{(2)}$ requires that

$$\partial_t \mathbf{Q}_+^{(0)} + \mathbf{v}_+^{(0)} \cdot \nabla \mathbf{Q}_+^{(0)} + \mathbf{Q}_+^{(0)} \cdot \boldsymbol{\Omega}_+^{(0)} - \boldsymbol{\Omega}_+^{(0)} \cdot \mathbf{Q}_+^{(0)} - \mathbf{S}_{\mathbf{Q}_+^{(0)}}(\mathbf{D}_+^{(0)}) - \Delta \mathbf{Q}_+^{(0)} \perp \ker f'(\mathbf{Q}_+^{(0)}). \quad (5.11)$$

Then by Lemma 3.2 in [26], (5.11) implies that $\mathbf{n}(t, x)$ should satisfy

$$\mathbf{n} \times \left(-s_+ \Delta \mathbf{n} + s_+ \mathbf{N} - \frac{\xi(2+s_+)}{3} \mathbf{D}_+^{(0)} \cdot \mathbf{n} \right) = 0,$$

where $\mathbf{N} = \frac{\partial \mathbf{n}}{\partial t} + \mathbf{v}_+^{(0)} \cdot \nabla \mathbf{n} - \boldsymbol{\Omega}_+^{(0)} \cdot \mathbf{n}$. And by Lemma 3.3 in [26], we know that

$$\begin{aligned} \sigma_{(0),+}^s + \sigma_{(0),+}^a &= \alpha_1(\mathbf{nn}) : \mathbf{D}_+^{(0)} \mathbf{nn} + \alpha_2 \mathbf{Nn} + \alpha_3 \mathbf{nN} + \alpha_4 \mathbf{D}_+^{(0)} + \alpha_5 \mathbf{D}_+^{(0)} \cdot \mathbf{nn} \\ &\quad + \alpha_6 \mathbf{nn} \cdot \mathbf{D}_+^{(0)} \triangleq \sigma^L, \\ \sigma_{(0),+}^d &= -2 \nabla \mathbf{n} \odot \nabla \mathbf{n} \triangleq \sigma^E, \end{aligned}$$

where $\alpha_1, \dots, \alpha_6$ are called the Leslie coefficients given by

$$\begin{aligned} \alpha_1 &= -\frac{2\xi^2 s_+^2 (3 - 2s_+)(1 + 2s_+)}{3}, \quad \alpha_2 = -s_+^2 - \frac{\xi s_+(2 + s_+)}{3}, \quad \alpha_3 = s_+^2 - \frac{\xi s_+(2 + s_+)}{3}, \\ \alpha_4 &= 1 + \frac{4\xi^2 (1 - s_+)^2}{9}, \quad \alpha_5 = \frac{\xi^2 s_+(4 - s_+)}{3} + \frac{\xi s_+(2 + s_+)}{3}, \quad \alpha_6 = \frac{\xi^2 s_+(4 - s_+)}{3} - \frac{\xi s_+(2 + s_+)}{3}. \end{aligned}$$

Moreover, we have

$$\sigma_{(0),-}^s + \sigma_{(0),-}^a + \sigma_{(0),-}^d = \mathbf{D}_-^{(0)}.$$

In summary, in the region $\Omega^+(t)$, $(\mathbf{v}_+^{(0)}, \mathbf{n})$ satisfies

$$\begin{aligned} \partial_t \mathbf{v}_+^{(0)} + \mathbf{v}_+^{(0)} \cdot \nabla \mathbf{v}_+^{(0)} &= -\nabla \mathbf{p}_+^{(0)} + \nabla \cdot (\sigma^L + \sigma^E), \\ \nabla \cdot \mathbf{v}_+^{(0)} &= 0, \\ \mathbf{n} \times (-\Delta \mathbf{n} + \mathbf{N} - \mathbf{D}_+^{(0)} \cdot \mathbf{n}) &= 0, \end{aligned}$$

which is the Ericksen-Leslie system introduced by Ericksen and Leslie [8, 13]. While in the region $\Omega^-(t)$, $\mathbf{v}_-^{(0)}$ satisfies the incompressible Navier-Stokes equations

$$\begin{aligned} \partial_t \mathbf{v}_-^{(0)} + \mathbf{v}_-^{(0)} \cdot \nabla \mathbf{v}_-^{(0)} &= -\nabla \mathbf{p}_-^{(0)} + \Delta \mathbf{v}_-^{(0)}, \\ \nabla \cdot \mathbf{v}_-^{(0)} &= 0. \end{aligned}$$

5.2. Inner expansion. Using the fact that

$$\nabla \cdot \sigma_\varepsilon^d = -\nabla \left(\frac{1}{2} |\nabla \mathbf{Q}^\varepsilon|^2 + \varepsilon^{-2} F_b(\mathbf{Q}^\varepsilon) \right) - \mathbf{H}^\varepsilon : \nabla \mathbf{Q}^\varepsilon,$$

the equation (1.3) can be rewritten as

$$\mathbf{v}_t^\varepsilon + \mathbf{v}^\varepsilon \nabla \mathbf{v}^\varepsilon - \Delta \mathbf{v}^\varepsilon + \nabla \mathbf{q}^\varepsilon = \nabla \cdot (\sigma_\varepsilon^a - \mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{H}^\varepsilon)) - \mathbf{H}^\varepsilon : \nabla \mathbf{Q}^\varepsilon, \quad (5.12)$$

where q^ε is the modified pressure defined by

$$q^\varepsilon = p^\varepsilon + \frac{1}{2} |\nabla \mathbf{Q}^\varepsilon|^2 + \varepsilon^{-2} F_b(\mathbf{Q}^\varepsilon).$$

In the transition region, we make the following expansion for $(\mathbf{v}^\varepsilon, \mathbf{p}^\varepsilon, \mathbf{Q}^\varepsilon)$:

$$\mathbf{v}^\varepsilon(t, x) = \tilde{\mathbf{v}}^{(0)}(\tau, t, x, z) + \varepsilon \tilde{\mathbf{v}}^{(1)}(\tau, t, x, z) + \dots, \quad (5.13)$$

$$\mathbf{p}^\varepsilon(t, x) = \varepsilon^{-2} \tilde{\mathbf{p}}^{(-2)}(\tau, t, x, z) + \varepsilon^{-1} \tilde{\mathbf{p}}^{(-1)}(\tau, t, x, z) + \dots, \quad (5.14)$$

$$\mathbf{Q}^\varepsilon(t, x) = \tilde{\mathbf{Q}}^{(0)}(\tau, t, x, z) + \varepsilon \tilde{\mathbf{Q}}^{(1)}(\tau, t, x, z) + \dots, \quad (5.15)$$

with $z = \frac{\varphi(t,x)}{\varepsilon}$. Then \mathbf{D}^ε and $\mathbf{\Omega}^\varepsilon$ has the expansion

$$\begin{aligned}\mathbf{D}^\varepsilon &= \varepsilon^{-1} \frac{1}{2} \left((\nabla \varphi \tilde{\mathbf{v}}_z^{(0)}) + (\nabla \varphi \tilde{\mathbf{v}}_z^{(0)})^T \right) + \frac{1}{2} \left((\nabla \varphi \tilde{\mathbf{v}}_z^{(1)}) + (\nabla \varphi \tilde{\mathbf{v}}_z^{(1)})^T \right) \\ &\quad + (\nabla_x \tilde{\mathbf{v}}^{(0)}) + (\nabla_x \tilde{\mathbf{v}}^{(0)})^T + \dots \triangleq \varepsilon^{-1} \tilde{\mathbf{D}}^{(-1)} + \tilde{\mathbf{D}}^{(0)} + \dots, \\ \mathbf{\Omega}^\varepsilon &= \varepsilon^{-1} \frac{1}{2} \left((\nabla \varphi \tilde{\mathbf{v}}_z^{(0)}) - (\nabla \varphi \tilde{\mathbf{v}}_z^{(0)})^T \right) + \frac{1}{2} \left((\nabla \varphi \tilde{\mathbf{v}}_z^{(1)}) - (\nabla \varphi \tilde{\mathbf{v}}_z^{(1)})^T \right) \\ &\quad + (\nabla_x \tilde{\mathbf{v}}^{(0)}) - (\nabla_x \tilde{\mathbf{v}}^{(0)})^T + \dots \triangleq \varepsilon^{-1} \tilde{\mathbf{\Omega}}^{(-1)} + \tilde{\mathbf{\Omega}}^{(0)} + \dots,\end{aligned}$$

The molecular field \mathbf{H}^ε has the expansion

$$\begin{aligned}\mathbf{H}^\varepsilon &= \varepsilon^{-2} \left(\tilde{\mathbf{Q}}_{zz}^{(0)} - f(\tilde{\mathbf{Q}}^{(0)}) \right) + \varepsilon^{-1} \left(\tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} \right. \\ &\quad \left. + \tilde{\mathbf{Q}}_z^{(0)} \Delta \varphi + 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(0)} \right) + \left(\tilde{\mathbf{Q}}_{zz}^{(2)} - f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(2)} - \frac{1}{2} \langle f''(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(1)} \rangle \right. \\ &\quad \left. + \tilde{\mathbf{Q}}_z^{(1)} \Delta \varphi + 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(1)} + \Delta_x \tilde{\mathbf{Q}}^{(0)} \right) + \dots \\ &\triangleq \varepsilon^{-2} \tilde{\mathbf{H}}^{(-2)} + \varepsilon^{-1} \tilde{\mathbf{H}}^{(-1)} + \tilde{\mathbf{H}}^{(0)} + \dots.\end{aligned}$$

The anti-symmetry viscous stress σ_ε^a has the expansion

$$\begin{aligned}\sigma_\varepsilon^a &= \varepsilon^{-2} \left(\tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{H}}^{(-2)} - \tilde{\mathbf{H}}^{(-2)} \cdot \tilde{\mathbf{Q}}^{(0)} \right) + \varepsilon^{-1} \left(\tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{H}}^{(-1)} - \tilde{\mathbf{H}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} \right. \\ &\quad \left. + \tilde{\mathbf{Q}}^{(1)} \cdot \tilde{\mathbf{H}}^{(-2)} - \tilde{\mathbf{H}}^{(-2)} \cdot \tilde{\mathbf{Q}}^{(1)} \right) + \left(\tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{H}}^{(0)} - \tilde{\mathbf{H}}^{(0)} \cdot \tilde{\mathbf{Q}}^{(0)} \right) + \dots \\ &\triangleq \varepsilon^{-2} \tilde{\sigma}_{(-2)}^a + \varepsilon^{-1} \tilde{\sigma}_{(-1)}^a + \tilde{\sigma}_{(0)}^a \dots.\end{aligned}$$

The modified pressure \tilde{q}^ε has the expansion

$$\begin{aligned}\tilde{q}^\varepsilon &= \varepsilon^{-2} \left(\frac{1}{2} |\tilde{\mathbf{Q}}_z^{(0)}|^2 + F_b(\tilde{\mathbf{Q}}^{(0)}) + \tilde{\mathbf{p}}^{(-2)} \right) + \varepsilon^{-1} \left(\tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}_z^{(1)} \right. \\ &\quad \left. + f(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)} + \nabla \varphi \cdot \nabla \tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)} + \tilde{\mathbf{p}}^{(-1)} \right) \\ &\quad + \left(\tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}_z^{(2)} + \frac{1}{2} |\tilde{\mathbf{Q}}_z^{(1)}|^2 + f(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(2)} + \frac{1}{2} \langle f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(1)} \rangle \right. \\ &\quad \left. + \nabla \varphi \cdot \nabla \tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(1)} + \nabla \varphi \cdot \nabla \tilde{\mathbf{Q}}^{(1)} : \tilde{\mathbf{Q}}_z^{(0)} + \frac{1}{2} \nabla_x \tilde{\mathbf{Q}}^{(0)} : \nabla_x \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{p}}^{(0)} \right) + \dots \\ &\triangleq \varepsilon^{-2} \tilde{\mathbf{q}}^{(-2)} + \varepsilon^{-1} \tilde{\mathbf{q}}^{(-1)} + \tilde{\mathbf{q}}^{(0)} + \dots.\end{aligned}$$

Now we plug these expansions into (5.12), then equate ε^k ($k = -3, -2, -1$) terms to obtain

$$\partial_z (\tilde{\sigma}_{-2}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-2)})) \cdot \nabla \varphi = \tilde{\mathbf{H}}^{(-2)} : \nabla \varphi \tilde{\mathbf{Q}}_z^{(0)} + \nabla \varphi \tilde{\mathbf{q}}_z^{(-2)}, \quad (5.16)$$

$$\begin{aligned}\frac{\nu}{2} \tilde{\mathbf{v}}_{zz}^{(0)} &= \varphi_\tau \tilde{\mathbf{v}}_z^{(0)} - \partial_z (\tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-2)})) \cdot \nabla \varphi \\ &\quad - \nabla \cdot (\tilde{\sigma}_{(-2)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-2)})) + \nabla \varphi \tilde{\mathbf{q}}_z^{(-1)} + \nabla_x \tilde{\mathbf{q}}^{(-2)} \\ &\quad + (\tilde{\mathbf{H}}^{(-2)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(0)}) \nabla \varphi + \tilde{\mathbf{H}}^{(-2)} : \nabla_x \tilde{\mathbf{Q}}^{(0)},\end{aligned} \quad (5.17)$$

and

$$\begin{aligned}
\tilde{\mathbf{v}}_{zz}^{(1)} = & \varphi_\tau \tilde{\mathbf{v}}_z^{(1)} - \partial_z (\tilde{\sigma}_{(0)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(2)}}(\tilde{\mathbf{H}}^{(-2)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-1)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(0)})) \cdot \nabla \varphi \\
& - \nabla_x \cdot (\tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-2)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) + \nabla \varphi \tilde{\mathbf{q}}_z^{(0)} + \nabla_x \tilde{\mathbf{q}}^{(-1)} \\
& + (\varphi_t \tilde{\mathbf{v}}_z^{(0)} + \tilde{\mathbf{v}}_\tau^{(0)} + \tilde{\mathbf{v}}^{(0)} \cdot \nabla \varphi \tilde{\mathbf{v}}_z^{(0)} - \tilde{\mathbf{v}}_z^{(0)} \Delta \varphi - 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{v}}_z^{(0)}) \\
& + (\tilde{\mathbf{H}}^{(-2)} : \tilde{\mathbf{Q}}_z^{(2)} + \tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) \nabla \varphi \\
& + (\tilde{\mathbf{H}}^{(-2)} : \nabla_x \tilde{\mathbf{Q}}^{(1)} + \tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}).
\end{aligned} \tag{5.18}$$

Equating ε^k ($k = -1, 0$) terms in (1.4), we obtain

$$\nabla \varphi \cdot \tilde{\mathbf{v}}_z^{(0)} = 0, \tag{5.19}$$

$$\nabla \varphi \cdot \tilde{\mathbf{v}}_z^{(1)} + \nabla_x \cdot \tilde{\mathbf{v}}^{(0)} = 0. \tag{5.20}$$

Equating ε^k ($k = -2, -1$) terms in (1.5), we obtain

$$-\varphi_\tau \tilde{\mathbf{Q}}_z^{(0)} + \tilde{\mathbf{Q}}_{zz}^{(0)} - f(\tilde{\mathbf{Q}}^{(0)}) = 0, \tag{5.21}$$

$$\begin{aligned}
\tilde{\mathbf{H}}^{(-1)} + \varphi_\tau \tilde{\mathbf{Q}}_z^{(1)} = & (\nabla \varphi \cdot \tilde{\mathbf{v}}^{(0)} + \varphi_t) \tilde{\mathbf{Q}}_z^{(0)} \\
& - \tilde{\mathbf{Q}}_\tau^{(0)} - \tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}} \cdot (\tilde{\mathbf{D}}^{(-1)}).
\end{aligned} \tag{5.22}$$

Multiplying (5.21) by $\tilde{\mathbf{Q}}^{(0)}$ and integrating from $z = -\infty$ to $z = +\infty$, we obtain

$$\varphi_\tau \int_{-\infty}^{\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz = F_b(\tilde{\mathbf{Q}}^{(0)}(\tau, t, x, -\infty)) - F_b(\tilde{\mathbf{Q}}^{(0)}(\tau, t, x, +\infty)) = 0. \tag{5.23}$$

Hence,

$$\varphi_\tau = 0, \quad -\tilde{\mathbf{Q}}_{zz}^{(0)} + f(\tilde{\mathbf{Q}}^{(0)}) = 0, \tag{5.24}$$

and $\tilde{\mathbf{H}}^{(-2)} = 0$ by (5.21). Hence,

$$\tilde{\sigma}_{-2}^a = 0, \quad \tilde{\sigma}_{-1}^a = \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{H}}^{(-1)} - \tilde{\mathbf{H}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)}, \tag{5.25}$$

and by (5.16),

$$\tilde{\mathbf{q}}_z^{(-2)} = \partial_z \left(\frac{1}{2} |\tilde{\mathbf{Q}}_z^{(0)}|^2 + F_b(\tilde{\mathbf{Q}}^{(0)}) + \tilde{\mathbf{p}}^{(-2)} \right) = 0. \tag{5.26}$$

Then the equation (5.17) is reduced to

$$\begin{aligned}
\tilde{\mathbf{v}}_{zz}^{(0)} = & -\partial_z (\tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) \cdot \nabla \varphi + \nabla \varphi \tilde{\mathbf{q}}_z^{(-1)} + \nabla_x \tilde{\mathbf{q}}^{(-2)} \\
& + \tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(0)} \nabla \varphi.
\end{aligned} \tag{5.27}$$

While, the equation (5.18) is reduced to

$$\begin{aligned}
\tilde{\mathbf{v}}_{zz}^{(1)} = & -\partial_z (\tilde{\sigma}_{(0)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-1)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(0)})) \cdot \nabla \varphi \\
& - \nabla_x \cdot (\tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) + \nabla \varphi \tilde{\mathbf{q}}_z^{(0)} + \nabla_x \tilde{\mathbf{q}}^{(-1)} \\
& + (\varphi_t \tilde{\mathbf{v}}_z^{(0)} + \tilde{\mathbf{v}}_\tau^{(0)} + \tilde{\mathbf{v}}^{(0)} \cdot \nabla \varphi \tilde{\mathbf{v}}_z^{(0)} - \tilde{\mathbf{v}}_z^{(0)} \Delta \varphi - 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{v}}_z^{(0)}) \\
& + (\tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) \nabla \varphi + \tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}.
\end{aligned} \tag{5.28}$$

5.3. Evolution of the sharp interface. By (5.24), it is reasonable to assume that $\tilde{\mathbf{Q}}^{(0)}(\tau, t, x, z) = s(z)(\mathbf{n}(t, x)\mathbf{n}(t, x) - \frac{1}{3}\mathbf{I})$, hence $\partial_\tau \tilde{\mathbf{Q}}^{(0)} = 0$. Thanks to the definition of $\tilde{\mathbf{H}}^{(-1)}$, the equation (5.22) can be rewritten as

$$\begin{aligned} & \tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)} \\ &= (\nabla\varphi \cdot \tilde{\mathbf{v}}^{(0)} + \varphi_t - \Delta\varphi - 2\nabla\varphi \cdot \nabla_x)\tilde{\mathbf{Q}}_z^{(0)} - \tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}} \cdot (\tilde{\mathbf{D}}^{(-1)}). \end{aligned} \quad (5.29)$$

Multiplying (5.29) by $\tilde{\mathbf{Q}}_z^{(0)}$ and integrating from $z = -\infty$ to $z = +\infty$, we obtain

$$\begin{aligned} \varphi_t &= \Delta\varphi - \nabla\varphi \cdot \mathbf{v}^{(0)} + \frac{\nabla\varphi \cdot \nabla \int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz}{\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz} \\ &\quad + \left(\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz \right)^{-1} \int_{-\infty}^{\infty} \left(\tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} - \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)} + \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{D}}^{(-1)}) \right) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &= \Delta\varphi - \nabla\varphi \cdot \mathbf{v}^{(0)} + \frac{\nabla\varphi \cdot \nabla \int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz}{\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz} \\ &\quad + \left(\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz \right)^{-1} \int_{-\infty}^{\infty} \tilde{\boldsymbol{\Omega}}^{(-1)} : (\tilde{\mathbf{Q}}_z^{(0)}\tilde{\mathbf{Q}}^{(0)} - \tilde{\mathbf{Q}}^{(0)}\tilde{\mathbf{Q}}_z^{(0)}) dz \\ &\quad + \xi \left(\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz \right)^{-1} \int_{-\infty}^{\infty} \tilde{\mathbf{D}}^{(-1)} : (\tilde{\mathbf{Q}}_z^{(0)}\tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)}\tilde{\mathbf{Q}}_z^{(0)}) dz \\ &\quad + \xi \left(\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz \right)^{-1} \int_{-\infty}^{\infty} \left(\tilde{\mathbf{D}}^{(-1)} : \frac{2}{3}\tilde{\mathbf{Q}}_z^{(0)} - 2(\tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) (\tilde{\mathbf{D}}^{(-1)} : \tilde{\mathbf{Q}}^{(0)}) \right) dz. \end{aligned} \quad (5.30)$$

Here we used the fact that $\nabla\varphi \cdot \tilde{\mathbf{v}}^{(0)} = \nabla\varphi \cdot \mathbf{v}^{(0)}$ by (5.19).

As $\int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz$ is independent of (t, x) , the equation (5.30) is reduced to

$$\begin{aligned} \varphi_t &= \Delta\varphi - \nabla\varphi \cdot \mathbf{v}^{(0)} \\ &\quad + \frac{\xi}{2} \left(\int_{-\infty}^{+\infty} |s'(z)|^2 dz \right)^{-1} \int_{-\infty}^{\infty} s'(1 + s - 2s^2) \tilde{\mathbf{D}}^{(-1)} : (\mathbf{nn}) dz. \end{aligned} \quad (5.31)$$

5.4. Jump conditions on the sharp interface. In this subsection, we will derive the jump condition of the velocity and the pressure. For this end, we assume that as $z \rightarrow \pm\infty$,

$$\begin{aligned} \tilde{\mathbf{v}}^{(k)}(\tau, t, x, z) &\rightarrow \mathbf{v}_{\pm}^{(k)}(\tau, t, x) \quad k = 0, 1, \\ \tilde{\mathbf{Q}}^{(k)}(\tau, x, z) &\rightarrow \mathbf{Q}_{\pm}^{(k)}(\tau, t, x) \quad k = 0, 1, \\ \tilde{\mathbf{p}}^{(k)}(\tau, t, x, z) &\rightarrow \mathbf{p}_{\pm}^{(k)}(\tau, t, x) \quad k = -2, -1, 0. \end{aligned}$$

We also assume that $\tilde{\mathbf{Q}}^{(0)}(\tau, t, x, z) = s(z)(\mathbf{n}(t, x)\mathbf{n}(t, x) - \frac{1}{3}\mathbf{I})$. In the sequel, ν is the unit normal to the sharp interface $\Gamma(t)$ and $[\cdot]$ denotes the jump across $\Gamma(t)$, i.e.,

$$[f] = f_+ - f_-|_{\Gamma(t)}.$$

First of all, integrating (5.19) from $z = -\infty$ to $z = +\infty$, we derive the jump condition of the velocity on $\Gamma(t)$:

$$[\mathbf{v}^{(0)}] \cdot \nu = 0 \quad \text{on} \quad \Gamma(t). \quad (5.32)$$

Integrating (5.26) from $z = -\infty$ to $z = +\infty$, we obtain

$$[\mathbf{p}^{(-2)}] = -[F_b(\mathbf{Q}^{(0)})] = 0 \quad \text{on} \quad \Gamma(t). \quad (5.33)$$

Note that $\nabla_x \tilde{\mathbf{q}}^{(-2)} = \nabla_x \mathbf{p}^{(-2)} = 0$. Multiplying (5.27) by $\nabla\varphi$, we get by (5.19) that

$$\tilde{\mathbf{q}}_z^{(-1)} = \partial_z \left\langle \tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)}), \nabla\varphi \otimes \nabla\varphi \right\rangle - \tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(0)}. \quad (5.34)$$

Then we get by integrating (5.34) from $z = -\infty$ to $z = +\infty$ that

$$\begin{aligned} [\mathbf{q}^{(-1)}] &= - \int_{-\infty}^{+\infty} \tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &= - (\Delta\varphi + \nabla\varphi \cdot \nabla) \int_{-\infty}^{+\infty} |\tilde{\mathbf{Q}}_z^{(0)}|^2 dz \\ &= - \frac{2}{3} \Delta\varphi \int_{-\infty}^{+\infty} |s'(z)|^2 dz, \end{aligned} \quad (5.35)$$

here we used the fact that

$$\begin{aligned} &\int_{-\infty}^{+\infty} \langle \tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}_z^{(0)} \rangle dz \\ &= \int_{-\infty}^{+\infty} \langle \tilde{\mathbf{Q}}_{zzz}^{(0)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}_z^{(0)}, \tilde{\mathbf{Q}}^{(1)} \rangle dz = 0 \quad \text{by (5.24)}. \end{aligned}$$

By (5.24) again, we have

$$\begin{aligned} \tilde{\mathbf{q}}^{(-1)} &= \tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}_z^{(1)} + f(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)} + \nabla\varphi \cdot \nabla\tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)} + \tilde{\mathbf{p}}^{(-1)} \\ &= \partial_z(\tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}^{(1)}) + \nabla\varphi \cdot \nabla\tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)} + \tilde{\mathbf{p}}^{(-1)}, \end{aligned} \quad (5.36)$$

which along with (5.35) implies that

$$[\mathbf{p}^{(-1)}] = [\mathbf{q}^{(-1)}] = -\frac{2}{3} \Delta\varphi \int_{-\infty}^{+\infty} |s'(z)|^2 dz. \quad (5.37)$$

Next we derive the jump condition of $\mathbf{p}_{\pm}^{(0)}$. We get by (5.19) that

$$\begin{aligned} &(\varphi_t \tilde{\mathbf{v}}_z^{(0)} + \tilde{\mathbf{v}}_{\tau}^{(0)} + \tilde{\mathbf{v}}^{(0)} \cdot \nabla\varphi \tilde{\mathbf{v}}_z^{(0)} - \tilde{\mathbf{v}}_z^{(0)} \Delta\varphi - 2\nabla\varphi \cdot \nabla_x \tilde{\mathbf{v}}_z^{(0)}) \cdot \nabla\varphi \\ &= \tilde{\mathbf{v}}_{\tau}^{(0)} \cdot \nabla\varphi - \langle 2\tilde{\mathbf{D}}^{(0)}, \nabla\varphi \otimes \nabla\varphi \rangle, \end{aligned}$$

and by (5.20), we have

$$\tilde{\mathbf{v}}_{zz}^{(1)} \cdot \nabla\varphi = -(\nabla_x \cdot \tilde{\mathbf{v}}^{(0)})_z.$$

Then multiply (5.28) by $\nabla\varphi$ to obtain

$$\begin{aligned} \tilde{\mathbf{q}}_z^{(0)} &= -(\nabla_x \tilde{\mathbf{q}}^{(-1)} - \nabla \mathbf{p}^{(-1)}_{\pm}) \cdot \nabla\varphi + (\nabla_x \cdot \tilde{\mathbf{v}}^{(0)})_z \\ &\quad + \partial_z \left\langle \tilde{\sigma}_{(0)}^a + 2\tilde{\mathbf{D}}^{(0)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-1)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}_0), \nabla\varphi \otimes \nabla\varphi \right\rangle \\ &\quad + \left(\nabla_x \cdot (\tilde{\sigma}_{(-1)}^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) \right) \cdot \nabla\varphi \\ &\quad - (\tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) - (\tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}) \cdot \nabla\varphi. \end{aligned} \quad (5.38)$$

Here we used the fact that

$$\nabla\varphi \cdot \tilde{\mathbf{v}}_{\tau}^{(0)} = (\nabla\varphi \cdot \mathbf{v}_{\pm}^{(0)})_{\tau} = -\nabla \mathbf{p}^{(-1)}_{\pm} \cdot \nabla\varphi.$$

Integrating (5.38) from $z = -\infty$ to $z = +\infty$, we get by (5.25) that

$$\begin{aligned}
[\mathbf{q}^{(0)}] &= \left[\left\langle \tilde{\sigma}_{(0)}^a + 2\tilde{\mathbf{D}}^{(0)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-1)}) - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}_0), \nabla\varphi \otimes \nabla\varphi \right\rangle \right] \\
&= - \int_{-\infty}^{+\infty} (\nabla_x \tilde{\mathbf{Q}}^{(-1)} - \nabla \mathbf{p}^{(-1)}_{\pm}) \cdot \nabla\varphi dz \\
&\quad + \int_{-\infty}^{+\infty} \left(\nabla_x \cdot (\tilde{\mathbf{Q}}^{(0)} \tilde{\mathbf{H}}^{(-1)} - \tilde{\mathbf{H}}^{(-1)} \tilde{\mathbf{Q}}^{(0)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) \right) \cdot \nabla\varphi dz \\
&\quad - \int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) + (\tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}) \cdot \nabla\varphi dz.
\end{aligned} \tag{5.39}$$

Thanks to the definition of $\tilde{\mathbf{q}}^{(0)}$, we get by (5.5) and (5.6) that

$$[\mathbf{q}^{(0)}] = \int_{-\infty}^{+\infty} \nabla_x \tilde{\mathbf{Q}}^{(0)} : \nabla_x \tilde{\mathbf{Q}}_z^{(0)} dz + [\mathbf{p}^{(0)}], \tag{5.40}$$

$$[\tilde{\sigma}_0^a - \mathbf{S}_{\tilde{\mathbf{Q}}^{(1)}}(\tilde{\mathbf{H}}^{(-1)})] = [\mathbf{Q}^{(0)} \cdot \mathbf{H}^{(0)} - \mathbf{H}^{(0)} \cdot \mathbf{Q}^{(0)}]. \tag{5.41}$$

And by (5.19), we find

$$\begin{aligned}
[\langle \mathbf{D}^{(0)}, \nabla\varphi \otimes \nabla\varphi \rangle] &= \frac{1}{2} \int_{-\infty}^{+\infty} \langle \nabla \tilde{\mathbf{v}}_z^{(0)} + (\nabla \tilde{\mathbf{v}}_z^{(0)})^T, \nabla\varphi \otimes \nabla\varphi \rangle dz \\
&= \int_{-\infty}^{+\infty} \langle \nabla \tilde{\mathbf{v}}_z^{(0)}, \nabla\varphi \otimes \nabla\varphi \rangle dz \\
&= \int_{-\infty}^{+\infty} \left(\nabla(\nabla\varphi \cdot \tilde{\mathbf{v}}_z^{(0)}) \nabla\varphi - \frac{1}{2} \tilde{\mathbf{v}}_z^{(0)} \cdot \nabla |\nabla\varphi|^2 \right) dz = 0.
\end{aligned} \tag{5.42}$$

Thanks to the definition of σ^d , we have

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left((\tilde{\mathbf{Q}}_z^{(0)} \Delta\varphi + 2\nabla_x \tilde{\mathbf{Q}}_z^{(0)} \nabla\varphi) : \nabla_x \tilde{\mathbf{Q}}^{(0)} \right) \cdot \nabla\varphi dz \\
&= \int_{-\infty}^{+\infty} \left(\tilde{\mathbf{Q}}_z^{(0)} \Delta\varphi : \nabla_x \tilde{\mathbf{Q}}^{(0)} \right) \cdot \nabla\varphi dz - \left\langle \sigma^d(\mathbf{Q}^{(0)}, \mathbf{Q}^{(0)}), \nabla\varphi \otimes \nabla\varphi \right\rangle \\
&= - \left\langle \sigma^d(\mathbf{Q}^{(0)}, \mathbf{Q}^{(0)}), \nabla\varphi \otimes \nabla\varphi \right\rangle.
\end{aligned} \tag{5.43}$$

Let

$$\begin{aligned}
\tilde{\mathbf{H}}_1^{(-1)} &= \tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \\
\tilde{\mathbf{H}}_1^{(0)} &= \tilde{\mathbf{Q}}_{zz}^{(2)} - f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(2)} - \frac{1}{2} \langle f''(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(1)} \rangle.
\end{aligned}$$

We find that

$$\int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}_1^{(0)} : \tilde{\mathbf{Q}}_z^{(0)} + \tilde{\mathbf{H}}_1^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)}) dz = 0 \tag{5.44}$$

by the following two facts

$$\begin{aligned}
&\int_{-\infty}^{+\infty} (f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}) : \tilde{\mathbf{Q}}_z^{(1)} dz + \frac{1}{2} \int_{-\infty}^{+\infty} \langle f''(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}, \tilde{\mathbf{Q}}^{(1)} \rangle : \tilde{\mathbf{Q}}_z^{(0)} dz \\
&= \int_{-\infty}^{+\infty} \partial_z \left(\frac{a}{2} |\tilde{\mathbf{Q}}^{(1)}|^2 - \frac{b}{2} (\tilde{\mathbf{Q}}^{(1)} \cdot \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{Q}}^{(1)}) : \tilde{\mathbf{Q}}^{(1)} + \frac{c}{2} |\tilde{\mathbf{Q}}^{(1)}|^2 |\tilde{\mathbf{Q}}^{(0)}|^2 + c (\tilde{\mathbf{Q}}^{(1)} : \tilde{\mathbf{Q}}^{(0)})^2 \right) dz \\
&= \frac{1}{2} \int_{-\infty}^{+\infty} \partial_z ((f'(\tilde{\mathbf{Q}}^{(0)}) \tilde{\mathbf{Q}}^{(1)}) : \tilde{\mathbf{Q}}^{(1)}) dz = \frac{1}{2} [(f'(\mathbf{Q}^{(0)}) \mathbf{Q}^{(1)}) : \mathbf{Q}^{(1)}] = 0,
\end{aligned}$$

and

$$\begin{aligned} & \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_{zz}^{(2)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(2)}) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &= \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_{zzz}^{(0)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}_z^{(0)}) : \tilde{\mathbf{Q}}^{(2)} dz = 0. \end{aligned}$$

Thanks to $\tilde{\mathbf{Q}}^{(0)} = s(z)(\mathbf{n}(t, x)\mathbf{n}(t, x) - \frac{1}{3}\mathbf{I})$, we find

$$\int_{-\infty}^{+\infty} \Delta_x \tilde{\mathbf{Q}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)} + \nabla_x \tilde{\mathbf{Q}}^{(0)} : \nabla_x \tilde{\mathbf{Q}}_z^{(0)} dz = 0. \quad (5.45)$$

Hence by (5.44) and (5.45), we have

$$\int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}^{(-1)} : \tilde{\mathbf{Q}}_z^{(1)} + \tilde{\mathbf{H}}^{(0)} : \tilde{\mathbf{Q}}_z^{(0)}) dz = \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_z^{(0)} \Delta \varphi + 2\nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) : \tilde{\mathbf{Q}}_z^{(1)}. \quad (5.46)$$

Summing up (5.39)–(5.43) and (5.46), we conclude that

$$\begin{aligned} & [\mathbf{p}^{(0)}] - \left[\left\langle 2\mathbf{D}^{(0)} + \mathbf{Q}^{(0)} \cdot \mathbf{H}^{(0)} - \mathbf{H}^{(0)} \cdot \mathbf{Q}^{(0)} - \mathbf{S}_{\mathbf{Q}^{(0)}}(\mathbf{H}^{(0)}) + \sigma^d(\mathbf{Q}^{(0)}, \mathbf{Q}^{(0)}), \nabla \varphi \otimes \nabla \varphi \right\rangle \right] \\ &= \int_{-\infty}^{+\infty} \left(\nabla_x \cdot (\tilde{\mathbf{Q}}^{(0)} \tilde{\mathbf{H}}^{(-1)} - \tilde{\mathbf{H}}^{(-1)} \tilde{\mathbf{Q}}^{(0)} - \mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{H}}^{(-1)})) \right) \cdot \nabla \varphi dz \\ &\quad - \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_z^{(0)} \Delta \varphi + 2\nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) : \tilde{\mathbf{Q}}_z^{(1)} + (\tilde{\mathbf{Q}}_z^{(1)} \Delta \varphi + 2\nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(1)}) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &\quad - \int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}) \cdot \nabla \varphi dz. \end{aligned} \quad (5.47)$$

By the calculations in section 4.1, the left hand side of (5.47) exactly equals to

$$[\mathbf{p}^{(0)}] - \left[\langle \sigma^L + \sigma^E, \nabla \varphi \otimes \nabla \varphi \rangle \right], \quad (5.48)$$

where $\sigma_-^L = \mathbf{D}^{(0)}$ and $\sigma_-^E = 0$.

In order that the velocity is continuous across the sharp interface, we take the constant $\xi = 0$ (see Remark 5.1). Then the evolution equation (5.31) of the sharp interface is reduced to

$$\varphi_t = \Delta \varphi - \mathbf{v}^{(0)} \cdot \nabla \varphi.$$

Hence, the equation (5.22) can be reduced to

$$\tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)} + 2\nabla \varphi \cdot \nabla \tilde{\mathbf{Q}}_z^{(0)} = -\tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)},$$

and

$$\begin{aligned} \tilde{\mathbf{H}}^{(-1)} &= \tilde{\mathbf{Q}}_z^{(0)} \Delta \varphi - \tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} + \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)}, \\ \tilde{\sigma}_{(-1)}^a &= \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\boldsymbol{\Omega}}^{(-1)} + \tilde{\boldsymbol{\Omega}}^{(-1)} \cdot \tilde{\mathbf{Q}}^{(0)} \cdot \tilde{\mathbf{Q}}^{(0)}. \end{aligned}$$

Then by (5.28), we get

$$\tilde{\mathbf{v}}_z^{(0)} = -\tilde{\sigma}_{(-1)}^a \cdot \nabla \varphi = \frac{s(z)}{2} \left((\mathbf{n} \cdot \nabla \varphi)^2 \tilde{\mathbf{v}}_z^{(0)} - (\mathbf{n} \cdot \tilde{\mathbf{v}}_z^{(0)}) (\mathbf{n} - \mathbf{n} \cdot \nabla \varphi \nabla \varphi) \right).$$

This implies that $\tilde{\mathbf{v}}_z^{(0)} = 0$. Hence, $[\mathbf{v}^{(0)}] = 0$ and

$$\tilde{\mathbf{Q}}_{zz}^{(1)} - f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)} + 2\nabla \varphi \cdot \nabla \tilde{\mathbf{Q}}_z^{(0)} = 0.$$

In a similar derivation to (4.21), we have

$$\nu \cdot \nabla \mathbf{n} = 0 \quad \text{on} \quad \Gamma(t), \quad (5.49)$$

which in turn implies

$$-\tilde{\mathbf{Q}}_{zz}^{(1)} + f'(\tilde{\mathbf{Q}}^{(0)})\tilde{\mathbf{Q}}^{(1)} = 0.$$

Thus, the solution $\tilde{\mathbf{Q}}^{(1)}$ may take

$$\tilde{\mathbf{Q}}^{(1)} = \partial_z \tilde{\mathbf{Q}}^{(0)} \quad \text{or} \quad \tilde{\mathbf{Q}}^{(1)} = s(z)(\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n}) \quad \mathbf{n}^\perp \in \mathbf{V}_\mathbf{n}, \quad (5.50)$$

or $\tilde{\mathbf{Q}}^{(1)} = \partial_z \tilde{\mathbf{Q}}^{(0)} + s(z)(\mathbf{n}\mathbf{n}^\perp + \mathbf{n}^\perp\mathbf{n})$.

Therefore, the jump of the pressure becomes

$$\begin{aligned} [\mathbf{p}^{(0)}] &= \left[\langle \sigma^L + \sigma^E, \nabla \varphi \otimes \nabla \varphi \rangle \right] \\ &= - \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_z^{(0)} \Delta \varphi + 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(0)}) : \tilde{\mathbf{Q}}_z^{(1)} + (\tilde{\mathbf{Q}}_z^{(1)} \Delta \varphi + 2 \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(1)}) : \tilde{\mathbf{Q}}_z^{(0)} dz \\ &\quad - \int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}) \cdot \nabla \varphi dz. \end{aligned}$$

By (5.49) and (5.50), we find

$$\begin{aligned} \left[\langle \sigma^E, \nabla \varphi \otimes \nabla \varphi \rangle \right] &= 0, \\ \int_{-\infty}^{+\infty} \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}_z^{(1)} + \nabla \varphi \cdot \nabla_x \tilde{\mathbf{Q}}_z^{(1)} : \tilde{\mathbf{Q}}_z^{(0)} dz &= 0, \\ \int_{-\infty}^{+\infty} (\tilde{\mathbf{H}}^{(-1)} : \nabla_x \tilde{\mathbf{Q}}^{(0)}) \cdot \nabla \varphi dz &= 0, \\ \int_{-\infty}^{+\infty} (\tilde{\mathbf{Q}}_z^{(0)} : \tilde{\mathbf{Q}}_z^{(1)}) \Delta \varphi + (\tilde{\mathbf{Q}}_z^{(1)} : \tilde{\mathbf{Q}}_z^{(0)}) \Delta \varphi dz &= 0. \end{aligned}$$

Then the jump condition of the pressure is reduced to

$$[\mathbf{p}^{(0)}] = \left[\langle \sigma^L, \nabla \varphi \otimes \nabla \varphi \rangle \right],$$

where

$$\sigma^L = s_+^2 \mathbf{n}\mathbf{N} - s_+^2 \mathbf{N}\mathbf{n} + \mathbf{D}^{(0)},$$

while the symmetric parts induced by $\mathbf{S}_{\mathbf{Q}^\varepsilon}(\mathbf{D}^\varepsilon)$ vanish.

Remark 5.1. In the case of the constant $\xi \neq 0$, $\tilde{\mathbf{v}}_z^{(0)} \neq 0$. Otherwise, we could conclude from (5.27) that

$$\mathbf{S}_{\tilde{\mathbf{Q}}^{(0)}}(\tilde{\mathbf{Q}}^{(0)}) = 0,$$

which is impossible. Therefore, the velocity is in general not continuous across the sharp interface in the case of $\xi \neq 0$ except that $\xi = O(\varepsilon)$.

In addition, the assumption on $\xi = 0$ also plays an important role for the global well-posedness of the 2-D Beris-Edwards system proved by Paicu and Zarnescu [21].

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